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Hamilton's turns and geometric phase for two-level systems

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Abstract. Hamilton, in the course of his studies on quaternions, introduced an elegant geometric representation for the composition of SU(2) elements, in terms of turns on the unit sphere S^2. We use these turns to study two-level systems, with particular reference to geometric phase. The special roles played by piecewise geodesic circuits in the state space and evolution under constant Hamiltonian are recognized.

1. Introduction

Two-level systems are by far the most intensively studied systems in the context of geometric phase, and they have played a major role in clarifying several aspects of this phase. The polarization optics of birefringent media and optically active media is an example of such a system. It is in this context that Pancharatnam [1] discovered one of the early examples of geometric phase [2, 3]. Since the pioneering work of Berry [4], a large number of geometric phase-related studies have been carried out in this area [5–14]. Geometric phase in the context of two-level systems has also been studied extensively in magnetic resonance [15–18], neutron interferometry [19–22] and two-level atoms [23–27].

For two-level systems the state space is $CP^1 = S^2$, and the group SU(2) plays a basic role in the dynamics of these systems. Traditionally one represents the state geometrically on S^2, but handles the composition of the SU(2) evolution operators algebraically. However, one can employ the elegant geometric construction of turns, introduced by Hamilton [28] in the context of his studies on quaternions, to handle the SU(2) evolution operators geometrically. (An excellent exposition of Hamilton's work and that of later authors can be found in the monograph of Biedenharn and Louck [29].) The power of such an approach in the context of several synthesis problems in polarization optics has been demonstrated recently [30–33]. These studies have also led, for the first time, to a generalization [34, 35] of Hamilton's ideas to the non-compact group $SU(1, 1) = SL(2, R) = Sp(2, R)$.

In the present paper we use Hamilton's turns to study two-level systems, particularly from the point of view of geometric phase. In section 2 we begin by recounting briefly how the elements of SU(2), as also their composition, can be represented geometrically using turns, and then show how turns act on the state space of two-level systems. In section 3 we explore in the context of geometric phase the special role played by

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piecewise geodesic closed circuits in the state space and evolutions under piecewise constant Hamiltonians. We then use Hamilton's turns to obtain insight into the geometric phase of two-level systems. For the purpose of clarity of presentation, we do this in three stages. We conclude in section 4 with some final remarks.

2. Hamilton's turns, two-level systems and the Poincaré–Bloch sphere

It is well known that matrices $u \in \mathrm{SU}(2)$ are determined by two complex (i.e. four real) parameters satisfying one real condition amongst them:

$$u = a_0 - \mathrm{i}\boldsymbol{a}\cdot\boldsymbol{\sigma} = \begin{pmatrix} a_0 - \mathrm{i}a_3 & -a_2 - \mathrm{i}a_1 \\ a_2 - \mathrm{i}a_1 & a_0 + \mathrm{i}a_3 \end{pmatrix} \qquad a_0^2 + \boldsymbol{a}\cdot\boldsymbol{a} = 1. \tag{2.1}$$

The a's are the homogeneous Euler parameters, determining a point on S^3. Hamilton's geometric notion of turns originates from the fact that u is equally well determined by an ordered pair of unit vectors $\hat{\boldsymbol{n}}, \hat{\boldsymbol{n}}' \in S^2$ such that the Euler parameters are

$$a_0 = \hat{\boldsymbol{n}}\cdot\hat{\boldsymbol{n}}' \qquad \boldsymbol{a} = \hat{\boldsymbol{n}} \wedge \hat{\boldsymbol{n}}'. \tag{2.2}$$

Now, the ordered pair $(\hat{\boldsymbol{n}}, \hat{\boldsymbol{n}}')$ can be pictured by the directed great circle arc on S^2 with the tail at $\hat{\boldsymbol{n}}$ and the head at $\hat{\boldsymbol{n}}'$. It is clear from (2.2) that u (i.e. a_0, $\boldsymbol{a}$) does not determine $\hat{\boldsymbol{n}}$, $\hat{\boldsymbol{n}}'$ uniquely; for, if $\hat{\boldsymbol{n}}$ and $\hat{\boldsymbol{n}}'$ are rotated by the same amount about the axis perpendicular to the great circle containing them, the resulting pair continues to satisfy (2.2) and hence determines the same u. Such a rotation corresponds, of course, to sliding the directed great circle arc representing u along its great circle.

Now, consider the equivalence classes of directed great circle arcs on S^2, the equivalence being with respect to the sliding discussed above. These equivalence classes are Hamilton's turns.

It is thus clear that turns and elements of SU(2) are in one-to-one correspondence. Thus, we can identify elements $u \in \mathrm{SU}(2)$ through the associated turns $A(\hat{\boldsymbol{n}}, \hat{\boldsymbol{n}}')$ as

$$u = A(\hat{\boldsymbol{n}}, \hat{\boldsymbol{n}}') = \hat{\boldsymbol{n}}\cdot\hat{\boldsymbol{n}}' - \mathrm{i}\hat{\boldsymbol{n}} \wedge \hat{\boldsymbol{n}}'\cdot\boldsymbol{\sigma} \tag{2.3}$$

and talk of turns and SU(2) elements interchangeably. It should always be remembered that for the turn $A(\hat{\boldsymbol{n}}, \hat{\boldsymbol{n}}')$ the representative directed great circle arc is from $\hat{\boldsymbol{n}}$ towards $\hat{\boldsymbol{n}}'$, through a length not exceeding π, with the tail at $\hat{\boldsymbol{n}}$ and the head at $\hat{\boldsymbol{n}}'$. For brevity, the representative arc itself will often be called the turn.

Apart from the one-to-one correspondence between turns and elements of SU(2), turns give to the SU(2) group operations a beautiful geometrical meaning. From (2.3) we see, by inspection, that

$$A(\hat{\boldsymbol{n}}, \hat{\boldsymbol{n}}')^{-1} = A(\hat{\boldsymbol{n}}, \hat{\boldsymbol{n}}')^{+} = A(\hat{\boldsymbol{n}}', \hat{\boldsymbol{n}}) \tag{2.4}$$

showing that the inverse of a turn is just the geometrically reversed turn. Further, for any two SU(2) elements u, u' with associated turns $A(\hat{\boldsymbol{n}}, \hat{\boldsymbol{n}}')$, $A(\hat{\boldsymbol{n}}', \hat{\boldsymbol{n}}'')$, respectively, the identity

$$\begin{aligned} u'u &= A(\hat{\boldsymbol{n}}', \hat{\boldsymbol{n}}'')A(\hat{\boldsymbol{n}}, \hat{\boldsymbol{n}}') \\ &= (\hat{\boldsymbol{n}}'\cdot\hat{\boldsymbol{n}}'' - \mathrm{i}\hat{\boldsymbol{n}}' \wedge \hat{\boldsymbol{n}}''\cdot\boldsymbol{\sigma})(\hat{\boldsymbol{n}}\cdot\hat{\boldsymbol{n}}' - \mathrm{i}\hat{\boldsymbol{n}} \wedge \hat{\boldsymbol{n}}'\cdot\boldsymbol{\sigma}) \\ &= \hat{\boldsymbol{n}}\cdot\hat{\boldsymbol{n}}'' - \mathrm{i}\hat{\boldsymbol{n}}\hat{\boldsymbol{n}}''\cdot\boldsymbol{\sigma} = A(\hat{\boldsymbol{n}}, \hat{\boldsymbol{n}}'') \end{aligned} \tag{2.5a}$$

brings out the following geometric meaning of SU(2) group multiplication. To compose two SU(2) elements u, u' *choose* their turns (using the freedom to slide along their respective great circles) such that the head of the right factor u and the tail of the left factor u' coincide (this is always possible, for any two great circles on S^2 either coincide or intersect). Then the turn corresponding to the SU(2) product $u'u$ is simply the great circle arc from the free tail of the right factor u to the free head of the left factor u'.

We will often find it convenient to write (2.5a) as

$$\text{turn } \hat{\boldsymbol{n}}\hat{\boldsymbol{n}}' + \text{turn } \hat{\boldsymbol{n}}'\hat{\boldsymbol{n}}'' = \text{turn } \hat{\boldsymbol{n}}\hat{\boldsymbol{n}}''. \tag{2.5b}$$

Note that when written this way the individual terms in the 'sum' read from left to right correspond to the factors in the SU(2) product read from right to left, so the order is important and the 'sum' is non-commutative.

This geometric 'addition' rule for turns is associative, and faithfully reproduces the non-commutative group composition in SU(2). It is reminiscent of the parallelogram law for the composition of elements of the Abelian Euclidean translation group.

The group SU(2), and hence Hamilton's theory of turns, is fundamental to the dynamics of two-level systems. The normalized states of a two-level system can be parametrized by unit vectors $\hat{\boldsymbol{n}} = (\cos\theta, \sin\theta\ \cos\varphi, \sin\theta\ \sin\varphi) \in S^2$, with θ and φ being the polar and azimuthal coordinates on S^2:

$$|\psi(\hat{\boldsymbol{n}})\rangle = \begin{pmatrix} \cos(\theta/2) \\ e^{i\varphi}\sin(\theta/2) \end{pmatrix}. \tag{2.6}$$

(This parametrization is well defined everywhere except at the 'south pole' $\theta = \pi$.) This state $|\psi(\hat{\boldsymbol{n}})\rangle$ leads to the 2×2 density matrix

$$\rho(\hat{\boldsymbol{n}}) = |\psi(\hat{\boldsymbol{n}})\rangle\langle\psi(\hat{\boldsymbol{n}})| = \tfrac{1}{2}(1 + \hat{\boldsymbol{n}}\cdot\boldsymbol{\sigma}). \tag{2.7}$$

With this one-to-one correspondence between states of the two-level system and points on S^2, the transformation of the state under an SU(2) evolution, i.e. under the action of a turn, is given by the familiar two-to-one SU(2)$\to$SO(3) homomorphism. For the particular case where the turn and the point representing the state are on the same great circle (the importance of which will soon become clear), we have

$$\begin{aligned} &A(\hat{\boldsymbol{n}}, \hat{\boldsymbol{n}}')\hat{\boldsymbol{n}}\cdot\boldsymbol{\sigma}A(\hat{\boldsymbol{n}}, \hat{\boldsymbol{n}}')^{-1} = \hat{\boldsymbol{n}}''\cdot\boldsymbol{\sigma} \\ &\hat{\boldsymbol{n}}'' = 2(\hat{\boldsymbol{n}}\cdot\hat{\boldsymbol{n}}')\hat{\boldsymbol{n}}' - \hat{\boldsymbol{n}} \\ &\hat{\boldsymbol{n}}\cdot\hat{\boldsymbol{n}}' = \hat{\boldsymbol{n}}'\cdot\hat{\boldsymbol{n}}'' \qquad \hat{\boldsymbol{n}}\wedge\hat{\boldsymbol{n}}' = -\hat{\boldsymbol{n}}''\wedge\hat{\boldsymbol{n}}'. \end{aligned} \tag{2.8}$$

Thus, the turn $A(\hat{\boldsymbol{n}}, \hat{\boldsymbol{n}}')$ acting on the state $\rho(\hat{\boldsymbol{n}})$, moves it along the great circle containing $\hat{\boldsymbol{n}}$ and $\hat{\boldsymbol{n}}'$, but past $\hat{\boldsymbol{n}}'$ to the point $\hat{\boldsymbol{n}}''$ which makes the same angle with $\hat{\boldsymbol{n}}'$ as $\hat{\boldsymbol{n}}$. Hence, the turn that will take $\rho(\hat{\boldsymbol{n}})$ to $\rho(\hat{\boldsymbol{n}}')$ is not $A(\hat{\boldsymbol{n}}, \hat{\boldsymbol{n}}')$ but 'half' of it, i.e. the turn $A(\hat{\boldsymbol{n}}, (\hat{\boldsymbol{n}}+\hat{\boldsymbol{n}}')/|\hat{\boldsymbol{n}}+\hat{\boldsymbol{n}}'|)$ with the tail at $\hat{\boldsymbol{n}}$ and the head at the mid-point of $\hat{\boldsymbol{n}}$ and $\hat{\boldsymbol{n}}'$:

$$A\left(\hat{\boldsymbol{n}}, \frac{\hat{\boldsymbol{n}}+\hat{\boldsymbol{n}}'}{|\hat{\boldsymbol{n}}+\hat{\boldsymbol{n}}'|}\right)\rho(\hat{\boldsymbol{n}})A\left(\hat{\boldsymbol{n}}, \frac{\hat{\boldsymbol{n}}+\hat{\boldsymbol{n}}'}{|\hat{\boldsymbol{n}}+\hat{\boldsymbol{n}}'|}\right)^{-1} = \rho(\hat{\boldsymbol{n}}'). \tag{2.9}$$

We have seen that states are points on S^2, whereas turns are (equivalence classes of) directed great circle arcs on S^2. To avoid confusion, it is useful to view them as two separate unit spheres. We can call the former the Poincaŕe–Bloch† sphere $\mathscr{P}$ and

† It is called the Poincaŕe sphere in polarization optics, and the Bloch sphere in the context of spin systems. In the case of two-level atoms one calls it the pseudo-Bloch sphere.

the latter the sphere of turns $\mathcal{T}$. From the fact that

$$\begin{aligned} &\hat{\boldsymbol{n}}\cdot\hat{\boldsymbol{n}}'-\mathrm{i}\hat{\boldsymbol{n}}\wedge\hat{\boldsymbol{n}}'\cdot\boldsymbol{\sigma}=\cos\frac{\alpha}{2}-\mathrm{i}\sin\frac{\alpha}{2}\hat{\boldsymbol{m}}\cdot\boldsymbol{\sigma} \\ &\frac{\alpha}{2}=\sin^{-1}|\hat{\boldsymbol{n}}\wedge\hat{\boldsymbol{n}}'| \qquad \hat{\boldsymbol{m}}=\frac{\hat{\boldsymbol{n}}\wedge\hat{\boldsymbol{n}}'}{|\hat{\boldsymbol{n}}\wedge\hat{\boldsymbol{n}}'|}\end{aligned} \tag{2.10}$$

we see the connection between turns and the axis-angle description of SU(2) elements:

$$A(\hat{\boldsymbol{n}},\hat{\boldsymbol{n}}')=\exp\left(-\mathrm{i}\frac{\alpha}{2}\hat{\boldsymbol{m}}\cdot\boldsymbol{\sigma}\right). \tag{2.11}$$

Here $\alpha/2$ is the angle (not exceeding π) between the unit vectors $\hat{\boldsymbol{n}}$ and $\hat{\boldsymbol{n}}'$, and $\hat{\boldsymbol{m}}$ is the unit vector in the direction of $\hat{\boldsymbol{n}}\wedge\hat{\boldsymbol{n}}'$. From (2.11) it is clear that the effect of a turn $A(\hat{\boldsymbol{n}},\hat{\boldsymbol{n}}')$ on the Poincaŕe–Bloch sphere $\mathcal{P}$ is to produce a right-handed rotation of amount $2\sin^{-1}|\hat{\boldsymbol{n}}\wedge\hat{\boldsymbol{n}}'|$ with $\hat{\boldsymbol{n}}\wedge\hat{\boldsymbol{n}}'$ as the axis of rotation. This rotation will transport points on $\mathcal{P}$ along circles of constant latitude with respect to $\hat{\boldsymbol{n}}\wedge\hat{\boldsymbol{n}}'$. Thus, states on the geodesic (great circle) perpendicular to $\hat{\boldsymbol{n}}\wedge\hat{\boldsymbol{n}}'$ on $\mathcal{P}$, and only these states, will trace geodesic trajectories under the action of the turn $A(\hat{\boldsymbol{n}},\hat{\boldsymbol{n}}')$.(The situation in (2.8), (2.9) clearly corresponds to this case.) It is this geodesic evolution that will be exploited in the next section to obtain the connection between turns and geometric phase.

3. Hamilton's turns and geometric phase

The state space of a system is the space of pure state density operators, i.e. the space of unit rays. It is also called the projective Hilbert space of the system, and the normalized state vectors constitute a U(1) bundle over this space. When a state is transported along a closed circuit in the state space in accordance with the quantum mechanical equation of motion, the final state vector would have, in general, picked up a phase relative to the initial vector. Part of this phase depends on the specific Hamiltonian used (out of a continuous family of possible choices) to transport the state along the circuit. This part is called the dynamical phase, and is simply given by the integral of the instantaneous expectation value of the Hamiltonian along the circuit [36]. The remaining part is the geometric phase, and depends only on the state space circuit. It is independent of the choice of Hamiltonian, and is invariant under re-parametrization of the circuit.

It follows that, given a circuit in the state space, if we transport a state along this circuit using a Hamiltonian which has zero expectation value at each point on the circuit, then the phase picked up by the state vector will be simply the geometric phase associated with this state space circuit, the dynamical phase being zero. This fact, which is important in the design of geometric phase experiments [6, 7], is the basis for the special status enjoyed by piecewise geodesic circuits: for every geodesic piece we can choose a *constant* Hamiltonian to transport a state vector along the piece with zero dynamical phase. Thus, piecewise geodesic circuits in state space and evolution under piecewise constant Hamiltonians are intimately connected.

It is this fact that we exploit to study the connection between Hamilton's theory of turns and the geometric phase of two-level systems. For convenience we do this in three stages.

3.1. *Geodesic triangle*

For a two-level system (polarization optics, spin system or two-level atom) the state space is the Poincaré–Bloch sphere $\mathscr{P}$. Consider on $\mathscr{P}$ a geodesic triangle circuit $\hat{\boldsymbol{n}}_1\hat{\boldsymbol{n}}_2\hat{\boldsymbol{n}}_3$ as shown in figure 1(a) where we have taken, without loss of generality, $\hat{\boldsymbol{n}}_1$ at the north pole. We will first present the piecewise constant Hamiltonian which produces evolution along this piecewise geodesic circuit. To this end, recall from (2.9) that $A(\hat{\boldsymbol{n}}_1,\ (\hat{\boldsymbol{n}}_1+\hat{\boldsymbol{n}}_2)/|\hat{\boldsymbol{n}}_1+\hat{\boldsymbol{n}}_2|)$ is an SU(2) element which acts on $\hat{\boldsymbol{n}}_1$ to evolve it to $\hat{\boldsymbol{n}}_2$. Further, since the term linear in $\boldsymbol{\sigma}$ in

$$A\left(\hat{\boldsymbol{n}}_1, \frac{\hat{\boldsymbol{n}}_1+\hat{\boldsymbol{n}}_2}{|\hat{\boldsymbol{n}}_1+\hat{\boldsymbol{n}}_2|}\right)=\frac{1+\hat{\boldsymbol{n}}_1\cdot\hat{\boldsymbol{n}}_2}{\sqrt{2(1+\hat{\boldsymbol{n}}_1\cdot\hat{\boldsymbol{n}}_2)}}-\frac{\mathrm{i}\hat{\boldsymbol{n}}_1\wedge\hat{\boldsymbol{n}}_2\cdot\boldsymbol{\sigma}}{\sqrt{2(1+\hat{\boldsymbol{n}}_1\cdot\hat{\boldsymbol{n}}_2)}} \tag{3.1}$$

is proportional to $\hat{\boldsymbol{n}}_1\wedge\hat{\boldsymbol{n}}_2$ it is clear that

$$A\left(\hat{\boldsymbol{n}}_1, \frac{\hat{\boldsymbol{n}}_1+\hat{\boldsymbol{n}}_2}{|\hat{\boldsymbol{n}}_1+\hat{\boldsymbol{n}}_2|}\right)=\exp(-\mathrm{i}\beta\hat{\boldsymbol{n}}\wedge\hat{\boldsymbol{n}}_2\cdot\boldsymbol{\sigma}) \tag{3.2}$$

for some real β. Indeed, β can be shown to equal $\frac{1}{2}\sin^{-1}|\hat{\boldsymbol{n}}_1\wedge\hat{\boldsymbol{n}}_2|$, but the actual value of β turns out to be unimportant for our argument. It is clear from (3.2) that $A(\hat{\boldsymbol{n}}_1, (\hat{\boldsymbol{n}}_1+\hat{\boldsymbol{n}}_2)/|\hat{\boldsymbol{n}}_1+\hat{\boldsymbol{n}}_2|)$ is a unitary evolution for unit time under the constant Hermitian Hamiltonian

$$H=\beta\hat{\boldsymbol{n}}_1\wedge\hat{\boldsymbol{n}}_2\cdot\boldsymbol{\sigma}. \tag{3.3}$$

Now, the states along the geodesic arc $\hat{\boldsymbol{n}}_1\hat{\boldsymbol{n}}_2$ on $\mathscr{P}$ are of the form

$$\begin{aligned}&\rho(\hat{\boldsymbol{n}}(\lambda))=\tfrac{1}{2}(1+\hat{\boldsymbol{n}}(\lambda)\cdot\boldsymbol{\sigma})\\ &\hat{\boldsymbol{n}}(\lambda)=\frac{(1-\lambda)\hat{\boldsymbol{n}}_1+\lambda\hat{\boldsymbol{n}}_2}{|(1-\lambda)\hat{\boldsymbol{n}}_1+\lambda\hat{\boldsymbol{n}}_2|}\qquad 0\leqslant\lambda\leqslant 1.\end{aligned} \tag{3.4}$$

It is clear that for each λ, and hence for all states along the geodesic arc $\hat{\boldsymbol{n}}_1\hat{\boldsymbol{n}}_2$, the expectation value of H, namely $\mathrm{tr}(\rho(\lambda)H)$, is zero. Thus, $A(\hat{\boldsymbol{n}}_1,\ (\hat{\boldsymbol{n}}+\hat{\boldsymbol{n}}_2)/|\hat{\boldsymbol{n}}_1+\hat{\boldsymbol{n}}_2|)$ represented by the turn $\hat{\boldsymbol{n}}_1\hat{\boldsymbol{m}}_3$ with $\hat{\boldsymbol{m}}_3=(\hat{\boldsymbol{n}}_1+\hat{\boldsymbol{n}}_2)/|\hat{\boldsymbol{n}}_1+\hat{\boldsymbol{n}}_2|$ on the sphere of turns $\mathscr{T}$ in figure 1(b) indeed transports the state $\hat{\boldsymbol{n}}_1$ to $\hat{\boldsymbol{n}}_2$ on $\mathscr{P}$ along the geodesic arc ensuring

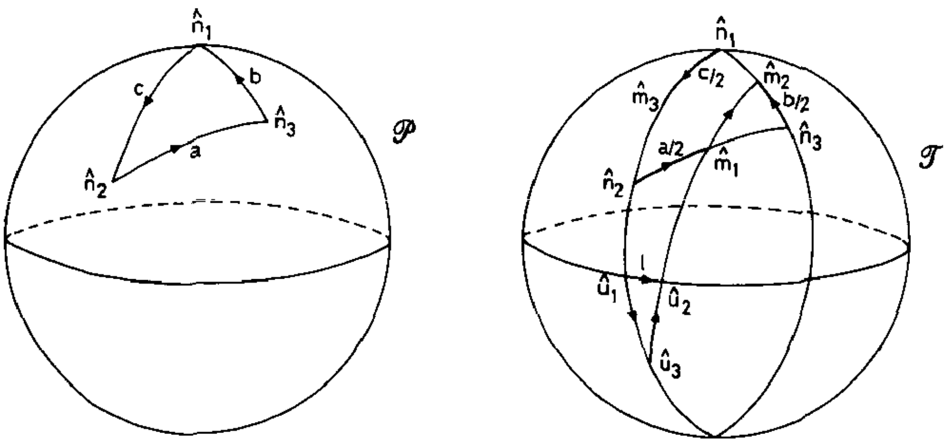

Figure 1. (a) Geodesic triangle circuit on the Poincaré sphere. (b) Piecewise constant Hamiltonians represented on the sphere of turns.

zero dynamical phase. Similarly, with $\hat{\boldsymbol{m}}_1=(\hat{\boldsymbol{n}}_2+\hat{\boldsymbol{n}}_3)/|\hat{\boldsymbol{n}}_2+\hat{\boldsymbol{n}}_3|$ and $\hat{\boldsymbol{m}}_2=(\hat{\boldsymbol{n}}_3+\hat{\boldsymbol{n}}_1)/|\hat{\boldsymbol{n}}_3+\hat{\boldsymbol{n}}_1|$ the turns $\hat{\boldsymbol{n}}_2\hat{\boldsymbol{m}}_1$ and $\hat{\boldsymbol{n}}_3\hat{\boldsymbol{m}}_2$ respectively transport the state along $\hat{\boldsymbol{n}}_2\hat{\boldsymbol{n}}_3$ and $\hat{\boldsymbol{n}}_3\hat{\boldsymbol{n}}_1$ with zero dynamical phase. Thus, the sequence of turns $\hat{\boldsymbol{n}}_1\hat{\boldsymbol{m}}_3$, $\hat{\boldsymbol{n}}_2\hat{\boldsymbol{m}}_1$, $\hat{\boldsymbol{n}}_3\hat{\boldsymbol{m}}_2$ on $\mathscr{T}$ transports the state $\hat{\boldsymbol{n}}_1$ back to itself along the geodesic triangle $\hat{\boldsymbol{n}}_1\hat{\boldsymbol{n}}_2\hat{\boldsymbol{n}}_3$ on $\mathscr{P}$; the Hamiltonian is piecewise constant and the dynamical phase is zero.

To make further progress we have to compose these three turns using the non-commutative, but associative, 'addition' rule for turns. We will show that

$$\text{turn } \hat{\boldsymbol{n}}_1\hat{\boldsymbol{m}}_3+\text{turn } \hat{\boldsymbol{n}}_2\hat{\boldsymbol{m}}_1+\text{turn } \hat{\boldsymbol{n}}_3\hat{\boldsymbol{m}}_2=\text{turn } \hat{\boldsymbol{u}}_1\hat{\boldsymbol{u}}_2. \tag{3.5}$$

Here $\hat{\boldsymbol{u}}$, $\hat{\boldsymbol{u}}_2$ are intersections of the (extensions of the) geodesics $\hat{\boldsymbol{n}}_1\hat{\boldsymbol{n}}_2$ and $\hat{\boldsymbol{m}}_2\hat{\boldsymbol{m}}_1$ with the equator as shown in figure 1(*b*).

To prove (3.5), first note that turn $\hat{\boldsymbol{n}}_2\hat{\boldsymbol{m}}_1=$ turn $\hat{\boldsymbol{m}}_1\hat{\boldsymbol{n}}_3$ and hence

$$\text{turn } \hat{\boldsymbol{n}}_2\hat{\boldsymbol{m}}_1+\text{turn } \hat{\boldsymbol{n}}_3\hat{\boldsymbol{m}}_2=\text{turn } \hat{\boldsymbol{m}}_1\hat{\boldsymbol{n}}_3+\text{turn } \hat{\boldsymbol{n}}_3\hat{\boldsymbol{m}}_2=\text{turn } \hat{\boldsymbol{m}}_1\hat{\boldsymbol{m}}_2 \tag{3.6}$$

by virtue of (2.5*b*). Further, if *a*, *b*, *c* are the (arc lengths of the) sides of the geodesic triangle on $\mathscr{P}$, then

$$\begin{gathered}\cos a=\hat{\boldsymbol{n}}_2\cdot\hat{\boldsymbol{n}}_3 \qquad \cos b=\hat{\boldsymbol{n}}_3\cdot\hat{\boldsymbol{n}}_1 \qquad \cos c=\hat{\boldsymbol{n}}_1\cdot\hat{\boldsymbol{n}}_2\\ \hat{\boldsymbol{m}}_1=\frac{\hat{\boldsymbol{n}}_2+\hat{\boldsymbol{n}}_3}{2\cos(a/2)} \qquad \hat{\boldsymbol{m}}_2=\frac{\hat{\boldsymbol{n}}_3+\hat{\boldsymbol{n}}_1}{2\cos(b/2)} \qquad \hat{\boldsymbol{m}}_3=\frac{\hat{\boldsymbol{n}}_1+\hat{\boldsymbol{n}}_2}{2\cos(c/2)}.\end{gathered} \tag{3.7}$$

Finally, note that $\hat{\boldsymbol{u}}_1$, $\hat{\boldsymbol{u}}_2$, $\hat{\boldsymbol{u}}_3$ can easily be expressed as linear combinations of $\hat{\boldsymbol{n}}_1$, $\hat{\boldsymbol{n}}_2$, $\hat{\boldsymbol{n}}_3$. Since $\hat{\boldsymbol{u}}_1$ is a linear combination of $\hat{\boldsymbol{n}}_1$ and $\hat{\boldsymbol{n}}_2$ and orthogonal to $\hat{\boldsymbol{n}}_1$ we have

$$\hat{\boldsymbol{u}}_1=\frac{\hat{\boldsymbol{n}}_2-\cos c\,\hat{\boldsymbol{n}}_1}{\sin c}. \tag{3.8}$$

Similarly, $\hat{\boldsymbol{u}}_3$ is a linear combination of $\hat{\boldsymbol{n}}_1$, $\hat{\boldsymbol{n}}_2$ as well as of $\hat{\boldsymbol{m}}_1$, $\hat{\boldsymbol{m}}_2$, hence it is orthogonal to $\hat{\boldsymbol{m}}_1\wedge\hat{\boldsymbol{m}}_2$; thus we find

$$\hat{\boldsymbol{u}}_3=\frac{\hat{\boldsymbol{n}}_2-\hat{\boldsymbol{n}}_1}{2\sin(c/2)}. \tag{3.9}$$

The fact that $\hat{\boldsymbol{u}}_2$ is a linear combination of $\hat{\boldsymbol{u}}_3$ and $\hat{\boldsymbol{m}}_2$ but orthogonal to $\hat{\boldsymbol{n}}_1$ leads to

$$\hat{\boldsymbol{u}}_2=\frac{\sin(c/2)(\hat{\boldsymbol{n}}_1+\hat{\boldsymbol{n}}_3)}{2\cos(a/2)\cos(b/2)}+\frac{\cos(b/2)(\hat{\boldsymbol{n}}_2-\hat{\boldsymbol{n}}_1)}{2\cos(a/2)\sin(c/2)}. \tag{3.10}$$

From (3.7)–(3.10) it is clear that $\hat{\boldsymbol{n}}_1\cdot\hat{\boldsymbol{m}}_3=\hat{\boldsymbol{u}}_1\cdot\hat{\boldsymbol{u}}_3$, $\hat{\boldsymbol{n}}_1\wedge\hat{\boldsymbol{m}}_3=\hat{\boldsymbol{u}}_1\wedge\hat{\boldsymbol{u}}_3$; and $\hat{\boldsymbol{m}}_1\cdot\hat{\boldsymbol{m}}_2=\hat{\boldsymbol{u}}_3\cdot\hat{\boldsymbol{u}}_2$, $\hat{\boldsymbol{m}}_1\wedge\hat{\boldsymbol{m}}_2=\hat{\boldsymbol{u}}_3\wedge\hat{\boldsymbol{u}}_2$, proving

$$\text{turn } \hat{\boldsymbol{n}}_1\hat{\boldsymbol{m}}_3=\text{turn } \hat{\boldsymbol{u}}_1\hat{\boldsymbol{u}}_3 \qquad \text{turn } \hat{\boldsymbol{m}}_1\hat{\boldsymbol{m}}_2=\text{turn } \hat{\boldsymbol{u}}_3\hat{\boldsymbol{u}}_2. \tag{3.11}$$

Combining (3.11) with (3.6), we have

$$\begin{aligned}&\text{turn } \hat{\boldsymbol{n}}_1\hat{\boldsymbol{m}}_3+\text{turn } \hat{\boldsymbol{n}}_2\hat{\boldsymbol{m}}_1+\text{turn } \hat{\boldsymbol{n}}_3\hat{\boldsymbol{m}}_2=\text{turn } \hat{\boldsymbol{n}}_1\hat{\boldsymbol{m}}_3+\text{turn } \hat{\boldsymbol{m}}_1\hat{\boldsymbol{m}}_2\\ &\qquad=\text{turn } \hat{\boldsymbol{u}}_1\hat{\boldsymbol{u}}_3+\text{turn } \hat{\boldsymbol{u}}_3\hat{\boldsymbol{u}}_2=\text{turn } \hat{\boldsymbol{u}}_1\hat{\boldsymbol{u}}_2\end{aligned} \tag{3.12}$$

which proves our assertion (3.5).

Having shown that the sum of the three turns is a turn on the great circle orthogonal to $\hat{\boldsymbol{n}}_1$, we have to compute its arc length *l*. Clearly $\cos l=\hat{\boldsymbol{u}}_1\cdot\hat{\boldsymbol{u}}_2$. Using the expressions for $\hat{\boldsymbol{u}}_1$, $\hat{\boldsymbol{u}}_2$ given in (3.8), (3.10) we can compute $\hat{\boldsymbol{u}}_1\cdot\hat{\boldsymbol{u}}_2$ and use (3.7) to cast the resulting expression in terms of *a*, *b*, *c*. This leads to the symmetric result

$$\cos l=\hat{\boldsymbol{u}}_1\cdot\hat{\boldsymbol{u}}_2=\frac{1+\cos a+\cos b+\cos c}{4\cos(a/2)\cos(b/2)\cos(c/2)}. \tag{3.13}$$

It can be immediately recognized that this is the standard expression for $\cos\frac{1}{2}\Delta(a, b, c)$, where $\Delta(a, b, c)$ is the area (solid angle) of a geodesic triangle with sides a, b, c. Thus, we see that the arc length l of the composed turn $\hat{\boldsymbol{u}}_1\hat{\boldsymbol{u}}_2$ is half the solid angle of the triangle along which the component turns cycle a state.

Since $\hat{\boldsymbol{u}}_1 \wedge \hat{\boldsymbol{u}}_2$ is in the direction of $\hat{\boldsymbol{n}}_1$ (the polar axis) we see that the unitary operator corresponding to the turn $\hat{\boldsymbol{u}}_1\hat{\boldsymbol{u}}_2$ is

$$A(\hat{\boldsymbol{u}}_1\hat{\boldsymbol{u}}_2) = \hat{\boldsymbol{u}}_1 \cdot \hat{\boldsymbol{u}}_2 - \mathrm{i}\hat{\boldsymbol{u}}_1 \wedge \hat{\boldsymbol{u}}_2 \cdot \boldsymbol{\sigma} = \cos\tfrac{1}{2}\Delta - \mathrm{i}\sin\tfrac{1}{2}\Delta\sigma_3 = \exp\left(-\mathrm{i}\frac{\Delta}{2}\cdot\sigma_3\right). \tag{3.14}$$

This unitary operator was constructed by composing the three individual unitary transformations which together take the state $\rho(\hat{\boldsymbol{n}}_1)$ back to itself along the geodesic triangle $\hat{\boldsymbol{n}}_1\hat{\boldsymbol{n}}_2\hat{\boldsymbol{n}}_3$. Consistent with this fact we see explicitly that the state vector

$$|\psi(\hat{\boldsymbol{n}}_1)\rangle \sim \begin{pmatrix}1\\0\end{pmatrix}. \tag{3.15}$$

Corresponding to $\hat{\boldsymbol{n}}_1$ on $\mathscr{P}$ picks up a phase $-\Delta/2$ under this evolution:

$$|\psi(\hat{\boldsymbol{n}}_1)\rangle \to \exp\left(-\mathrm{i}\frac{\Delta}{2}\sigma_3\right)|\psi(\hat{\boldsymbol{n}}_1)\rangle = \mathrm{e}^{-\mathrm{i}\Delta/2}|\psi(\hat{\boldsymbol{n}}_1)\rangle. \tag{3.16}$$

Since we have ensured that the dynamical phase is zero, this phase $-\Delta/2$ is simply the geometric phase. We have thus proved, using Hamilton's turns, that the geometric phase associated with a geodesic triangle on $\mathscr{P}$ equals minus half the area of the triangle.

It should be stressed that our results have invariant meaning, and are independent of our simplifying assumptions. For instance, we derived our results choosing counterclockwise (positive) sense for the circuit on $\mathscr{P}$. Now suppose the circuit had clockwise sense. Then the 'sum' turn in (3.5) would have reversed and as a consequence the (geometric) phase picked up by $\psi(\hat{\boldsymbol{n}}_1)$ would have reversed signature, as can be seen from (3.16). However, the relationship between area and geometric phase remains unchanged, for reversing the sense of the circuit changes the signature of its area. Similarly, we took $\hat{\boldsymbol{n}}_1$ to be along the polar axis. Since SU(2) acts transitively on $\mathscr{P}$, it is clear that if we take $\hat{\boldsymbol{n}}_1$ in some arbitrary direction the 'sum' turn would lie on the great circle orthogonal to $\hat{\boldsymbol{n}}_1$, will have an arc length equal to half the area of the triangle and will have, as viewed from $\hat{\boldsymbol{n}}_1$, the same sense as the circuit. That is, (3.16) is valid for an arbitrary direction of $\hat{\boldsymbol{n}}_1$: given $\hat{\boldsymbol{n}}_1$, under the action of a turn lying on the great circle orthogonal to $\hat{\boldsymbol{n}}_1$ the state vector $\psi(\hat{\boldsymbol{n}}_1)$ picks a phase with magnitude equal to the arc length of the turn and signature opposite to the sense of the turn as viewed from $\hat{\boldsymbol{n}}_1$.

3.2. Geodesic polygon

Having shown that for a geodesic triangle the geometric phase equals minus half the area of the triangle, it is now easy to show that this connection between geometric phase and area applies to an N-sided geodesic polygon as well.

Given a geodesic polygon $\hat{\boldsymbol{n}}_1\hat{\boldsymbol{n}}_2\hat{\boldsymbol{n}}_3\ldots\hat{\boldsymbol{n}}_N$ on $\mathscr{P}$ as shown in figure 2, the N turns, namely turn $\hat{\boldsymbol{n}}_1(\hat{\boldsymbol{n}}_1+\hat{\boldsymbol{n}}_2)/|\hat{\boldsymbol{n}}_1+\hat{\boldsymbol{n}}_2|$, turn $\hat{\boldsymbol{n}}_2(\hat{\boldsymbol{n}}_2+\hat{\boldsymbol{n}}_3)/|\hat{\boldsymbol{n}}_2+\hat{\boldsymbol{n}}_3|, \ldots,$ turn $\hat{\boldsymbol{n}}_{N-1}(\hat{\boldsymbol{n}}_{N-1}+\hat{\boldsymbol{n}}_N)/|\hat{\boldsymbol{n}}_{N-1}+\hat{\boldsymbol{n}}_N|$, turn $\hat{\boldsymbol{n}}_N(\hat{\boldsymbol{n}}_N+\hat{\boldsymbol{n}}_1)/|\hat{\boldsymbol{n}}_N+\hat{\boldsymbol{n}}_1|$, acting in that sequence will transport $\hat{\boldsymbol{n}}_1$ back to itself along the polygon, ensuring that the dynamical phase is zero (at every point on the circuit). It follows that to find the geometric phase associated with the polygon we have to simply find the 'sum' of these N turns, which we know corresponds

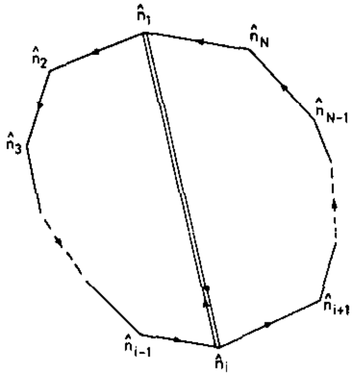

Figure 2. Geodesic polygon on the Poincaré sphere, with individual geodesic arcs drawn for simplicity as straight lines.

to the product of the unitary transformations with piecewise constant Hamiltonian 'acting along' the sides of the polygon. Let this sum be S.

Clearly, we can introduce (i.e. 'add') the null turn†

$$\text{turn } \hat{\boldsymbol{n}}_i\hat{\boldsymbol{n}}_i = \text{turn } \hat{\boldsymbol{n}}_i \frac{\hat{\boldsymbol{n}}_i + \hat{\boldsymbol{n}}_1}{|\hat{\boldsymbol{n}}_i + \hat{\boldsymbol{n}}_1|} + \text{turn } \hat{\boldsymbol{n}}_1 \frac{\hat{\boldsymbol{n}}_1 + \hat{\boldsymbol{n}}_i}{|\hat{\boldsymbol{n}}_1 + \hat{\boldsymbol{n}}_i|} \tag{3.17}$$

between turn $\hat{\boldsymbol{n}}_{i-1}(\hat{\boldsymbol{n}}_{i-1} + \hat{\boldsymbol{n}})/|\hat{\boldsymbol{n}}_{i-1} + \hat{\boldsymbol{n}}_i|$ and turn $\hat{\boldsymbol{n}}_i(\hat{\boldsymbol{n}}_i + \hat{\boldsymbol{n}}_{i+1})/|\hat{\boldsymbol{n}}_i + \hat{\boldsymbol{n}}_{i+1}|$, $i = 3, 4, \ldots, N-1$ in the sum without affecting it. That (3.17) is indeed a resolution of the null turn follows from the fact that turn $\hat{\boldsymbol{n}}_1(\hat{\boldsymbol{n}}_1 + \hat{\boldsymbol{n}}_i)/|\hat{\boldsymbol{n}}_1 + \hat{\boldsymbol{n}}_i|$ equals turn $[(\hat{\boldsymbol{n}}_i + \hat{\boldsymbol{n}}_1)/|\hat{\boldsymbol{n}}_i + \hat{\boldsymbol{n}}_1|]\hat{\boldsymbol{n}}_i$. This added null turn corresponds to going from $\hat{\boldsymbol{n}}_i$ to $\hat{\boldsymbol{n}}_1$ along the geodesic $\hat{\boldsymbol{n}}_i\hat{\boldsymbol{n}}_1$ and returning back to $\hat{\boldsymbol{n}}_i$ along this geodesic. With these $(N-3)$ resolutions of the null turn inserted in the original sum S of N turns, we have to now sum $N-2(N-3) = 3(N-2)$ turns.

Using associativity, we can now write this sum as a sum of $(N+2)$ partial sums, each corresponding to a sum of three turns, as follows:

$$S = \sum_{i=2}^{N-1}\left(\text{turn } \hat{\boldsymbol{n}}_1 \frac{\hat{\boldsymbol{n}}_1 + \hat{\boldsymbol{n}}_i}{|\hat{\boldsymbol{n}}_1 + \hat{\boldsymbol{n}}_i|} + \text{turn } \hat{\boldsymbol{n}}_i \frac{\hat{\boldsymbol{n}}_i + \hat{\boldsymbol{n}}_{i+1}}{|\hat{\boldsymbol{n}}_i + \hat{\boldsymbol{n}}_{i+1}|} + \text{turn } \hat{\boldsymbol{n}}_{i+1} \frac{\hat{\boldsymbol{n}}_{i+1} + \hat{\boldsymbol{n}}_1}{|\hat{\boldsymbol{n}}_{i+1} + \hat{\boldsymbol{n}}_1|}\right). \tag{3.18}$$

We will see that these $(N-2)$ partial sums commute with each other.

It is readily realized that the ith partial sum in (3.18) is just the sum of the three turns transporting $\hat{\boldsymbol{n}}_1$ back to itself along the ith geodesic *triangle* $\hat{\boldsymbol{n}}_1\hat{\boldsymbol{n}}_i\hat{\boldsymbol{n}}_{i+1}$ and ensures zero dynamical phase. And from our analysis in section 3.1 we know that this partial sum is a turn which lies in the great circle orthogonal to $\hat{\boldsymbol{n}}_1$, has arc length equal to half the area of the ith triangle $\hat{\boldsymbol{n}}_1\hat{\boldsymbol{n}}_i\hat{\boldsymbol{n}}_{i+1}$ and, as viewed from $\hat{\boldsymbol{n}}_1$, has the same sense as the triangle (and hence as the polygon).

It should be stressed that *both the great circle on which the ith partial sum turn lies and also the sense of this turn are independent of i.* Consequently, we see first that the sum of $(N-2)$ partial sums is an *Abelian* sum (they are on the same great circle) and then that the arc lengths of these $(N-2)$ partial sums simply add (they have the same sense) to give the arc length of S. Thus, S is a turn with arc length of amount $|\Delta|/2$ where Δ is the area of the polygon, it lies in the great circle orthogonal to $\hat{\boldsymbol{n}}_1$ and, as viewed from $\hat{\boldsymbol{n}}_1$, has the same sense as the polygon. It follows that the state $\psi(\hat{\boldsymbol{n}}_1)$ is an eigenstate of S with eigenvalue $e^{-i\Delta/2}$. This phase of minus half the solid angle picked up by $\psi(\hat{\boldsymbol{n}}_1)$ on its tour around the polygon is purely geometrical since it was ensured that the dynamical phase was zero.

† The null turn corresponds to the identity element of SU(2).

We have thus proved that the geometric phase associated with an N-sided polygon on $\mathscr{P}$ equals minus half of its area, for every N. The key element in this proof was the introduction of the $(N-3)$ resolutions of the null turn which divided the polygon into $(N-2)$ triangles all having a common vertex $\hat{n}_1$, rendering the geometric phase of the polygon to be simply the sum of the geometric phases of the $(N-2)$ triangles. It converted the original sum into a sum of commuting turns.

3.3. Arbitrary closed circuit

The above result for the N-sided geodesic polygon can be easily extended to an arbitrary closed circuit on $\mathscr{P}$ in an obvious manner.

Given a closed circuit on $\mathscr{P}$, we can approximate it by an N-sided geodesic polygon for sufficiently large N. Noting that the relationship between the geometric phase and area of an N-sided polygon is valid for all N, we can now take the limit $N \to \infty$ to see that the geometric phase associated with any closed circuit on $\mathscr{P}$ equals minus half of its area.

4. Concluding remarks

We have studied two-level systems and their geometric phases using Hamilton's turns. Our approach was algebraic, using geodesic polygons to develop the relation between geometric phase and area on $\mathscr{P}$; this brought out the importance of geodesics on the one hand and piecewise constant Hamiltonians on the other. Alternatively, one could have set up, in the spirit of [30], a differential equation connecting increments in the sum of turns to changes in the geometric phase as the circuit is deformed by infinitesimal amounts. But we believe that the present approach is more illuminating, with turns giving insight into the geometry of phases and geometric phase giving insight into the geometry of turns.

If we add the turns forming the sides of a directed piecewise geodesic polygon on $\mathscr{T}$ we indeed get the null turn; but if we take 'half' of these turns and then add them, we do not get the null turn. Instead we get a turn whose arc length equals half the area of the polygon, as was shown in section 3. The same arc length equals the geometric phase by construction (since the construction ensured that the dynamical phase was zero). This situation can be contrasted to translation vectors and directed polygons in Euclidean space, where taking half of the sides of the polygon will lead to a 'similar' polygon and hence these half vectors will again add to the null vector. It is the curvature of $\mathscr{T} = S^2$ which is responsible for the anholonomy of the half turns not adding up to the null turn. It is the same curvature of $\mathscr{P} = S^2$, sometimes viewed as a monopole at the origin, which is responsible for non-vanishing geometric phase.

The general methods and approach of the present paper may be fruitfully compared with the treatment of the geometric phase given in [37], based on quantum kinematic ideas. In particular, the important roles of geodesics in state space, and polygons, may be noted.

Finally, as noted in the introduction, Hamilton's turns have been generalized to the non-compact group SU(1, 1), in which context these geometric objects are called screws [34, 35]. Since there is considerable interest in the geometric phase of SU(1, 1) systems [38–43], it will be of considerable value to extend the present analysis to screws and systems with SU(1, 1) symmetry. We plan to return to this problem elsewhere.

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